

Minimum-energy broadcast in random-grid ad-hoc networks: approximation and distributed algorithms*

Tiziana Calamoneri
Dipartimento di Informatica
Sapienza Università di Roma
calamo@di.uniroma1.it

Andrea E.F. Clementi
Dipartimento di Matematica
Università di Roma II, "Tor
Vergata"
clementi@mat.uniroma2.it

Angelo Monti
Dipartimento di Informatica
Sapienza Università di Roma
monti@di.uniroma1.it

Gianluca Rossi
Dipartimento di Matematica
Università di Roma II, "Tor
Vergata"
gianluca.rossi@uniroma2.it

Riccardo Silvestri
Dipartimento di Informatica
Sapienza Università di Roma
silver@di.uniroma1.it

ABSTRACT

The MIN ENERGY BROADCAST problem consists in assigning transmission ranges to the nodes of an ad-hoc network in order to guarantee a directed spanning tree from a given *source* node and, at the same time, to minimize the energy consumption (i.e. the *energy cost*) yielded by the range assignment. MIN ENERGY BROADCAST is known to be NP-hard.

We consider *random-grid networks* where nodes are chosen independently at random from the n points of a $\sqrt{n} \times \sqrt{n}$ square grid in the plane. The probability of the existence of a node at a given point of the grid does depend on that point, that is, the probability distribution can be *non-uniform*.

By using information-theoretic arguments, we prove a lower bound $(1 - \epsilon)\frac{n}{\pi}$ on the energy cost of any feasible solution for this problem. Then, we provide an efficient solution of energy cost not larger than $1.1204\frac{n}{\pi}$.

Finally, we present a fully-distributed protocol that constructs a broadcast range assignment of energy cost not larger than $8n$, thus still yielding constant approximation. The *energy load* is well balanced and, at the same time, the *work complexity* (i.e. the energy due to all message transmissions of the protocol) is asymptotically optimal. The completion time of the protocol is only an $O(\log n)$ factor slower than the optimum. The approximation quality of our distributed solution is also experimentally evaluated.

All bounds hold with probability at least $1 - 1/n^{\Theta(1)}$.

1. INTRODUCTION

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Range assignments in ad-hoc networks. In ad-hoc networks, nodes are able to vary their transmission ranges in order to provide good network connectivity and low energy consumption at the same time. More precisely, the transmission ranges determine a (directed) *communication graph* $G(S, E)$ over the set S of nodes: a node v , with transmission range r , can transmit to another node w (so, edge $(u, w) \in E$) if and only if w belongs to the *disk* centered in v and of radius r . The transmission range of a node depends, in turn, on the energy power supplied to the node. In particular, the power P_v required by a node v to correctly transmit data to another station w must satisfy the inequality $P_v \geq \text{dist}(v, w)^2$, where $\text{dist}(v, w)$ is the Euclidean distance between v and w . In several works [1, 12, 17, 24], it is assumed that nodes can arbitrarily vary their transmission range over the "large" set $\{\text{dist}(s, t) | s, t \in S\}$. However, in some important network scenarios (like sensor networks), this assumption is not realistic: the adopted technology allows nodes to have only *few* possible transmission range values. For this reason, we adopt the model considered in [9, 10, 16, 38] where nodes are able to choose their transmission range from a *restricted* set Γ .

A fundamental class of algorithmic problems arising from ad-hoc wireless networks consists in the *range assignment problems*: find a transmission range assignment $r : S \rightarrow \Gamma$ such that (1) the resulting communication graph satisfies a given connectivity property Π , and (2) the *energy cost* $\text{cost}(r) = \sum r(s)^2$ of the assignment is minimized (see [17, 24]).

Several research works [1, 12, 8, 17] have been devoted to the MIN ENERGY BROADCAST problem where Π is defined as follows: *Given a source node s , the communication graph has to contain a directed spanning tree rooted at s (a branching from s).* Previous theoretical results on MIN ENERGY BROADCAST concern *worst-case* analysis only. This problem is known to be NP-hard [12] even when $\Gamma = \{0, l_1, l_2\}$ for $l_1 < l_2$ and l_1 is set to any fixed positive constant. The most famous approximation algorithm is the *MST*-based heuristic [17]. This heuristics works in $\Theta(n^2)$ time and its performance analysis has been the subject of several works over the last years [12, 18, 36]. In [1], it is finally proved

the *tight* bound 6 on its approximation ratio. More recently, a new polynomial-time algorithm is provided in [8] that achieves approximation ratio close to 4. This algorithm applies a rather complex edge-contraction technique on the MST-based solution. Its present best version works in $\Theta(n^5)$ time and the design of any efficient distributed version seems to be a very hard task.

It is important to observe that the MST-based heuristic is "far" from achieving optimal solutions even on a complete square *grid* of n points [5, 19]: its worst-case approximation ratio on such grids is not smaller than 3. In [19], it is also experimentally observed that this heuristic has a bad behavior when applied to random *regular* instances such as *faulty* square grids. Furthermore, the MST-based heuristic requires a large range set Γ .

The above discussion leads us to study MIN ENERGY BROADCAST over *random grid* networks. Given a $\sqrt{n} \times \sqrt{n}$ grid of points of the Euclidean plane (without loss of generality, adjacent points are placed at unit distance), each point¹ i is selected as a *node* of the random grid network independently with probability p_i . This *node probability* can be any value in the interval $[p_{min}, p_{max}]$ where p_{min} and p_{max} are two arbitrary positive constants in the interval $(0, 1)$. We remark that our random grids are in general non uniform: Random grids provide a good model for several ad-hoc and sensor networks. On one hand, by varying the p_i 's values, it is possible to model non homogenous input configurations with regions of different node densities. On the other hand, the grid structure guarantees a minimal distance among nodes: this is often a desired property in order to optimize area coverage and avoid message collisions. Nevertheless, as discussed later, all our results also hold for the standard *uniform random distribution* (i.e. the random input formed by choosing n points independently and uniformly at random from a 2-dimensional square) [25, 33].

Our results. 1) We provide a lower bound on the energy cost of feasible solutions for *any* range assignment problem on random grids where the required property Π *implies* the existence of a disk cover. We say that a range assignment is a (*disk*) *covering* assignment if it guarantees that every node of the network is within the *positive* range of some node. MIN ENERGY BROADCAST is just one of those important cases requiring covering range assignments.

Let l_1 be the minimum positive range in Γ . For any $0 < \epsilon < 1$, if $l_1 = \Omega(\frac{1}{\epsilon})$ then we prove that the energy cost of any covering range assignment is with high probability² (in short, w.h.p.) at least $(1 - \epsilon)\frac{n}{\pi}$. Observe that the lower bound tends to n/π for any l_1 such that $l_1 = \omega(1)$, so for minimal ranges much smaller than the *connectivity threshold* $\Theta(\sqrt{\log n})$ [15, 22, 33, 34].

The proof's technique of the lower bound departs significantly from all those adopted in this topic and uses information-theoretic arguments. By using this result, we will prove that

the next two algorithms are almost optimal.

2) We provide a simple and efficient algorithm for random grids that uses minimal range $l_1 = \Theta(\sqrt{\log n})$ and returns a solution of energy cost not larger than $1.1204\frac{n}{\pi}$ w.h.p.: In virtue of our lower bound, this is very close to the optimum. Observe that our lower bound holds for any covering range assignment while the upper bound holds for feasible range assignments of MIN ENERGY BROADCAST: this implies that, for $l_1 = \omega(1)$, the extra-cost, due to the required tree connectivity property, is "almost" negligible in random grids (it is still an open problem whether this is in fact negligible). Our algorithmic solution works in $O(n \log n)$ time and needs a set Γ of *logarithmic* size (in n). The range assignment is inspired to the one provided in [5] for complete square grids (i.e. every point of the grid is a node of the network). However, the probabilistic cost analysis of our construction for *random* grids is definitely not related to that in [5].

3) It is common opinion that the development of efficient, provably-good *distributed* algorithms is presently the major challenge about range assignment problems [4, 36, 17]. We provide an efficient distributed algorithm for MIN ENERGY BROADCAST on random grids. We investigate the performance of the protocol in two different scenarios: *single broadcast* and *many-broadcast*, i.e., a sequence of consecutive broadcast operations. In both cases, besides the energy cost of the returned range assignment, we consider further important complexity aspects that determine the quality of a distributed solution.

- *Work Complexity.* In the ad-hoc network model, the *work complexity* of a distributed algorithm (i.e. protocol) for MIN ENERGY BROADCAST is defined as the sum of the energy cost of *all* transmissions made by the protocol to perform the broadcast operation [23, 27, 35]. This complexity measure thus considers both the cost to *construct* the range assignment and the cost to *use* it to broadcast the message (the latter being exactly the cost of the range assignment defined for centralized algorithms). Since *both* the above energy costs are paid by the nodes, a protocol can be really considered energy efficient only if it has a small work complexity.

- *Energy-Load Balancing.* In some real ad-hoc networks (such as sensor networks), it is also important to equally distribute the energy load to all nodes. For instance, solutions, assigning large ranges to few nodes, are not feasible in scenarios where nodes have limited battery charges. In such a case, we aim to design solutions that are well energy-load balanced. Notice that, in the many-broadcast scenario, this corresponds to maximize *network lifetime* according to the model in [6, 7].

- (*Amortized*) *Completion Time.* Another relevant aspect of a broadcast protocol is the *completion time*, i.e., the number of time steps required to complete one broadcast operation. In the many-broadcast scenario, the *amortized completion time* is the *average* completion time for one broadcast operation.

Our aim is to derive a protocol having *provably-good* performance with respect to *all* the above complexity aspects. To the best of our knowledge, no available protocol has been shown to have this overall performance.

We first define a very simple range assignment where only

¹For the sake of simplicity, we here assume that points are labelled with index $i = 1, \dots, n$.

²Here and in the sequel the term *with high probability* means that the event holds with probability at least $1 - \frac{1}{n^a}$ for some constant $a > 0$.

one positive range in Γ is used, provided that it is not smaller than $c\sqrt{\log n}$ where c is a suitable positive constant (observe again that this value is asymptotically equivalent to the connectivity threshold). This solution is then shown to be w.h.p. feasible and to have an energy cost not larger than $8n$. Thanks to our lower bound, the achieved energy cost yields a constant approximation ratio. Moreover, this simple range assignment can be constructed and managed by an efficient protocol. We assume every node initially knows n and its relative position with respect to the grid only. Positioning information can be obtained by using GPS systems or Ad-Hoc Positioning System (APS) [31]. This assumption is reasonable in *static* ad-hoc networks since every node can store *once and for all* its position in the set-up phase. The protocol exploits a fully-distributed pivot-election strategy borrowed from [6].

We prove that the work complexity of the protocol is equivalent to the energy cost of the centralized version and hence, thanks again to our lower bound (clearly, a lower bound for the energy cost is also a lower bound for the work complexity), it achieves a constant approximation ratio as well. It is important to emphasize that the best distributed algorithm to compute an MST in the ad-hoc model has an expected work complexity $\Omega(n \log n)$ [21, 23, 28, 29]. By comparing this bound with the $\Theta(n)$ work complexity achieved by our protocol, we can state that any MST-based solution [17, 8] cannot yield good work complexity in this scenario. Other distributed solutions have been considered in the literature [4, 27, 35, 37], however their performance analysis is based on experimental tests only. We also compared the work complexity of our protocol to the energy cost of the centralized MST-based solution over thousands of random instances with different sizes and densities. The average performance ratio between the two solutions is always between 2 and 3 (see Section 4.1) thus confirming our analytical results.

Our protocol yields a good energy-load balanced solution: there are $\Theta(n/\log n)$ *pivots*, i.e., the nodes having range $\Theta(\sqrt{\log n})$ (the remaining nodes have range 0). Furthermore, thanks to the pivot-election strategy [6], a good energy-load balance is also obtained with respect to an arbitrary sequence of broadcast operations, i.e., for the many-broadcast scenario. At every new operation, the pivot task is indeed assigned to nodes according to a Round Robin rule. We show this yields an almost optimal *life-time* of the network according to the energy consumption model in [6, 7].

As for the single broadcast scenario, the completion time of our protocol is slower than the optimum by a $\Theta(\log n)$ factor. As for the many-broadcast scenario, when the number of broadcast operations is $\Omega(\log n)$, then the amortized completion time is optimal.

Finally, we notice that, by using the technique in [30], our protocol can be emulated on the standard uniform random distribution [25, 33]. The same holds for the centralized algorithm achieving cost $1.1204\frac{n}{\pi}$ and for the lower bound as well. The relative proofs for the uniform distribution are easier and, so, they are omitted in this extended abstract.

Paper's Organization. In Section 2, we provide the proof of the lower bound. In Section 3, we describe the centralized algorithm yielding almost optimal cost. Finally, Section 4 is

devoted to the description of our distributed protocol and its analysis.

1.1 Preliminaries

The square grid R of $\sqrt{n} \times \sqrt{n}$ points will be indexed from 1 to n . Without loss of generality, the distance between adjacent points is set to 1. To each point i of R , a probability value p_i is assigned such that $p_{min} \leq p_i \leq p_{max}$ where p_{min} and p_{max} are arbitrary constants in $(0, 1)$. We consider the random input model $\mathcal{R}(R, p_1, \dots, p_n)$ where an instance $S \subseteq R$ has probability

$$P_R(S) = \prod_{i \in S} p_i \cdot \prod_{i \in R-S} (1 - p_i)$$

Observe that this probability distribution is equivalent to select each point $i \in R$ independently with probability p_i . A selected point will be called *node*. In the sequel, a subset $S \subseteq R$ selected according to the above random distribution will be simply called *random set*.

A set of disks $\mathcal{C} = \{D_1, \dots, D_m\}$ is said to be an l -cover for $S \subseteq R$ if the following properties hold: *i)* All disks of \mathcal{C} have radius at least l , where l is some positive value. *ii)* Every disk of \mathcal{C} has its center on a node of S . *iii)* Every node of S is covered by some disk of \mathcal{C} .

Observe that a range assignment $r : S \rightarrow \Gamma = \{0, l_1, l_2, \dots, l_k\}$ can be represented by the family of disks $\mathcal{C} = \{C_1, \dots, C_\ell\}$ yielded by the positive values of r , and its energy cost is defined as

$$\text{cost}(\mathcal{C}) = \sum_{i=1}^{\ell} r_i^2 \quad \text{where } r_i \text{ is the radius of } C_i \quad (1)$$

Furthermore, a feasible range assignment for the MIN ENERGY BROADCAST problem, with input S and Γ , uniquely determines an l_1 -cover for S having the same cost. Notice that the converse is not true in general.

2. THE LOWER BOUND

In this section, we provide a lower bound on the cost of any *covering* range assignment for a random set $S \subseteq R$.

Definition 1 Let $\Pr[R, \epsilon, l]$ be the probability that a random set $S \subseteq R$ has an l -cover of cost not larger than $(1 - \epsilon)\frac{n}{\pi}$.

Theorem 2 Let δ, p_{min} and p_{max} be three constants such that $0 < \delta < 1$ and $0 < p_{min} \leq p_{max} < 1$. Let $S \subseteq R$ be any random set. Then, for any ϵ with $0 < \epsilon < 1$, for sufficiently large n , and for

$$l \geq \frac{5(1 - \epsilon)^{\frac{1}{6}}}{\epsilon(1 - \delta)p_{min}}$$

it holds that

$$\Pr[R, \epsilon, l] \leq 2^{-\frac{1}{100}(1 - \delta)\epsilon p_{min} n + \log(4n) + t} + e^{-\frac{\delta^2}{2} p_{min} \lfloor \frac{n}{t} \rfloor + t}$$

$$\text{where } t = \left\lceil \frac{100}{3} \frac{\log \left(\frac{1-p_{\min}}{p_{\min}} \frac{p_{\max}}{1-p_{\max}} \right) + 1}{\epsilon(1-\delta)p_{\min}} \right\rceil$$

The above theorem clearly implies our lower bound stated in the Introduction and it requires no restriction about the transmission-range set Γ but a lower bound on $l_1 = l$ that does not depend on n . In particular, if ϵ is *any* positive constant then, for sufficiently large grids and a sufficiently large *constant* l (so l does not depend on n), $\mathbf{Pr}[R, \epsilon, l]$ is not larger than the inverse of an exponential function in n .

The theorem's proof makes use of the following combinatorial result.

Lemma 3 *Let R_1, \dots, R_t be a partition of the n points in R and let (k_1, \dots, k_t) be any t -tuple of integers such that $0 \leq k_j \leq |R_j| = n_j$. Then, the number of subsets S of R such that $|S \cap R_j| = k_j$ ($1 \leq j \leq t$) admitting an l -cover \mathcal{C} with $l \geq \sqrt{\epsilon}$ and $\text{cost}(\mathcal{C}) \leq \frac{(1-\epsilon)n}{\pi}$ is at most*

$$2^{\lambda(n, q, \epsilon, l, t)} \cdot \prod_{j=1}^t \binom{n_j}{k_j}$$

where $q = \min \left\{ \frac{k_j}{n_j} \mid 1 \leq j \leq t \right\}$ and $\lambda(n, q, \epsilon, l, t) =$

$$\left(-q \log e \left(1 - (1-\epsilon) \left(1 + \frac{1}{2l^2} + \frac{1}{\sqrt{2}l} \right)^2 \right) + \frac{1-\epsilon}{\pi l^2} \log \frac{64e\pi l^6}{1-\epsilon} \right) n + \log(4n) + t$$

We now provide a brief description of the *information-theoretic* approach adopted to prove the above lemma.

Let S be a subset of points of R satisfying the hypothesis of the lemma. By exploiting the l -cover \mathcal{C} , we will prove that S can be encoded into a binary string $\text{cod}(S)$ of length at most

$$\log \left(2^{\lambda(n, q, \epsilon, l, t)} \cdot \prod_{j=1}^t \binom{n_j}{k_j} \right)$$

The lemma thus follows since the number of these sets S cannot exceed the number of binary strings of the above length.

Proof of Lemma 3. Let S be a subset of points of R satisfying the hypothesis of the lemma. Consider the l -cover \mathcal{C}' of S having the same centers of \mathcal{C} and where each radius r in \mathcal{C} is replaced with a radius $r' = \sqrt{\lceil r^2 \rceil}$. Clearly, this change is negligible in terms of cost.

We now show that, thanks to \mathcal{C}' , S can be encoded into a binary string of length at most

$$\log \left(2^{\lambda(n, q, \epsilon, l, t)} \cdot \prod_{j=1}^t \binom{n_j}{k_j} \right)$$

Thus the thesis follows by noting that the number of these sets S cannot exceed the number of binary strings of the above length.

The binary string $\text{cod}(S)$ encoding S is the concatenation of four substrings NUM , CEN , RAD and COV .

- a) NUM reports the number m of centers of \mathcal{C} .
- b) CEN reports information to recover the indices of the m nodes of S that are centers in \mathcal{C} (we assume that the n points in the grid are numbered from 1 to n).
- c) RAD reports information to recover the radii of the m nodes in \mathcal{C}' .
- d) COV reports information to recover the indices of the nodes in S .

We now explain how these data are encoded and then bound the length of each of the four substrings.

- a) The number m of centers in \mathcal{C} is at most $|S| \leq n$. Thus we encode it by a binary string of fixed length (i.e. $\lceil \log(n+1) \rceil$). Hence

$$|NUM| = \lceil \log(n+1) \rceil \leq \log n + 1 \quad (2)$$

- b) The centers of \mathcal{C} are a subset of the n points in R and so we encode them by a string of fixed length, i.e.

$$\left\lceil \log \left(\binom{n}{m} + 1 \right) \right\rceil$$

Since in the cover, each of the m centers has radius at least l , it must hold $ml^2 \leq \text{cost}(\mathcal{C})$. From the hypothesis $\text{cost}(\mathcal{C}) \leq (1-\epsilon)\pi n$, we get

$$m \leq \frac{(1-\epsilon)n}{\pi l^2} \quad (3)$$

As for substring CEN , we obtain

$$\begin{aligned} |CEN| &= \left\lceil \log \binom{n}{m} + 1 \right\rceil \leq \log \binom{n}{m} + 1 \\ &\leq m \log \frac{\epsilon n}{m} + 1 \leq \frac{(1-\epsilon)n}{\pi l^2} \log \frac{e\pi l^2}{1-\epsilon} + 1 \end{aligned}$$

Observe that in the above inequalities we used

$$\left(\frac{\epsilon n}{m} \right)^m \leq \left(\frac{e\pi l^2}{1-\epsilon} \right)^{\frac{(1-\epsilon)n}{\pi l^2}}$$

since the function $\left(\frac{\epsilon n}{x} \right)^x$ is increasing in the range $[1, n]$; then, from (3), m is in the range $[1, \frac{(1-\epsilon)n}{\pi l^2}]$.

- c) Let now r'_1, r'_2, \dots, r'_m be the radii in \mathcal{C}' arranged by increasing order of the indices of the m centers. In order to give the information on the radii of \mathcal{C}' , we encode string $\lceil r_1 \rceil^2 \# \lceil r_2 \rceil^2 \# \dots \lceil r_m \rceil^2 \#$ in binary where bit 0 is encoded as 00, bit 1 as 11 and the symbol $\#$ as 01.

We thus get

$$\begin{aligned}
|RAD| &= 2m + 2 \sum_{i=1}^m [\log(\lceil r_i^2 \rceil + 1)] \\
&\leq 2m + 2 \sum_{i=1}^m [\log(r_i^2 + 2)] \\
&\leq 2m + 2 \sum_{i=1}^m (\log r_i^2 + 2) \\
&= 2 \log \prod_{i=1}^m r_i^2 + 6m \leq 2 \log \left(\frac{\text{cost}(C)}{m} \right)^m + 6m \\
&\leq 2 \log l^{2 \frac{\text{cost}(C)}{l^2}} + 6m \leq 4 \frac{(1-\epsilon)n}{\pi l^2} \log l + 6 \frac{(1-\epsilon)n}{\pi l^2} \\
&= 4 \frac{(1-\epsilon)n}{\pi l^2} \log(\sqrt{8}l) \tag{4}
\end{aligned}$$

In the above inequalities, we first used $\prod_{i=1}^m r_i^2 \leq \left(\frac{\text{cost}(C)}{m} \right)^m$ since the product is maximized when all factors have the same value. Next, we used $\left(\frac{\text{cost}(C)}{m} \right)^m \leq l^{2 \frac{\text{cost}(C)}{l^2}}$ since the function $\left(\frac{\text{cost}(C)}{x} \right)^x$ is increasing in the range $[1, \frac{\text{cost}(C)}{e}]$; the value of m is in the range $[1, \text{cost}(C)/l^2]$; $ml^2 \leq \text{cost}(C)$ and $\text{cost}(C)/l^2 \leq \text{cost}(C)/e$ for $l \geq \sqrt{e}$. Finally, we bounded m using (3).

- d) In order to encode the nodes in S , we use t strings. The j -th string encodes the k_j points of S in R_j . The k_j points of S in R_j covered by C' are a subset of the n'_j points in R_j covered by C' . Hence, we encode these k_j points with a binary string of length

$$\left\lceil \log \prod_{j=1}^t \binom{n'_j}{k_j} + 1 \right\rceil$$

Since for integers a, b and c where $a \leq b \leq c$ it holds that $\binom{b}{a} \leq \left(\frac{b}{c} \right)^a \binom{c}{a}$, we get

$$\begin{aligned}
|COV| &= \sum_{j=1}^t \left\lceil \log \left(\binom{n'_j}{k_j} + 1 \right) \right\rceil \leq \sum_{j=1}^t \left(\log \binom{n'_j}{k_j} + 1 \right) \\
&\leq \log \prod_{j=1}^t \binom{n'_j}{k_j} + t \leq \log \left(\prod_{j=1}^t \left(\frac{n'_j}{n_j} \right)^{k_j} \binom{n_j}{k_j} \right) + t \\
&\leq \log \prod_{j=1}^t \left(\frac{n'_j}{n_j} \right)^{k_j} + \log \prod_{j=1}^t \binom{n_j}{k_j} + t \\
&= \log \prod_{j=1}^t \left(1 - \frac{n_j - n'_j}{n_j} \right)^{k_j} + \log \prod_{j=1}^t \binom{n_j}{k_j} + t \\
&\leq \log \prod_{j=1}^t e^{-\frac{k_j}{n_j} (n_j - n'_j)} + \log \prod_{j=1}^t \binom{n_j}{k_j} + t \\
&\leq \log e^{-\sum_{j=1}^t q(n_j - n'_j)} + \log \prod_{j=1}^t \binom{n_j}{k_j} + t \\
&= -q(n - n') \log e + \log \prod_{j=1}^t \binom{n_j}{k_j} + t \tag{5}
\end{aligned}$$

where n' is the number of points of R covered by C' . We now give an upper bound for n' .

Let x be the number of points of R covered by a disk D of radius r and, for each of these points, consider the square of area 1 centered in the point. These t squares are disjoint and are covered by the disk obtained by extending the radius r of disk D to $r + 1/\sqrt{2}$. So, the number of points of R covered by a disk of radius r is bounded by $\pi(r + 1/\sqrt{2})^2$. Moreover, it holds that

$$r'_i = \sqrt{\lceil r_i^2 \rceil} \leq \sqrt{r_i^2 + 1} < r_i + \frac{1}{2l}$$

where the last inequality follows since $r_i \geq l$. We thus obtain

$$\begin{aligned}
n' &\leq \sum_{i=1}^m \pi \left(r'_i + \frac{1}{\sqrt{2}} \right)^2 \leq \sum_{i=1}^m \pi \left(r_i + \frac{1}{2l} + \frac{1}{\sqrt{2}} \right)^2 \\
&= \pi \left(\sum_{i=1}^m r_i^2 + \left(\frac{1}{2l} + \frac{1}{\sqrt{2}} \right)^2 m + 2 \left(\frac{1}{2l} + \frac{1}{\sqrt{2}} \right) \sum_{i=1}^m r_i \right)
\end{aligned}$$

We can now use Hölder's inequality and obtain

$$\sum_{i=1}^m r_i = \sum_{i=1}^m (r_i^2)^{\frac{1}{2}} \leq m \left(\frac{\sum_{i=1}^m r_i^2}{m} \right)^{\frac{1}{2}} = \sqrt{m \cdot \text{cost}(C)}$$

Since $m \leq \text{cost}(C)/l^2$, we get

$$\begin{aligned}
n' &\leq \pi \text{cost}(C) \left(1 + \left(\frac{1}{2l} + \frac{1}{\sqrt{2}} \right)^2 \frac{1}{l^2} + 2 \left(\frac{1}{2l} + \frac{1}{\sqrt{2}} \right) \frac{1}{l} \right) \\
&\leq (1-\epsilon) \left(1 + \frac{1}{2l^2} + \frac{1}{\sqrt{2}l} \right)^2 n
\end{aligned}$$

From (5), we get

$$\begin{aligned}
|COV| &\leq -q \log e \left(1 - (1-\epsilon) \left(1 + \frac{1}{2l^2} + \frac{1}{\sqrt{2}l} \right)^2 \right) n \\
&\quad + \log \prod_{j=1}^t \binom{n'_j}{k_j} + t \tag{6}
\end{aligned}$$

We now combine the bounds in (2), (4), (4) and (6) (respectively on the lengths of NUM, CEN, RAD and COV) and obtain

$$\begin{aligned}
|\text{cod}(S)| &\leq \log n + 1 + \frac{(1-\epsilon)n}{\pi l^2} \log \frac{e\pi l^2}{1-\epsilon} + 1 \\
&\quad + 4 \frac{(1-\epsilon)n}{\pi l^2} \log(\sqrt{8}l) + \log \prod_{j=1}^t \binom{n'_j}{k_j} + t \\
&\quad - q \log e \left(1 - (1-\epsilon) \left(1 + \frac{1}{2l^2} + \frac{1}{\sqrt{2}l} \right)^2 \right) n \\
&= \log(4n) + \log \prod_{i=1}^t \binom{n_i}{k_i} + t + \left(\frac{1-\epsilon}{\pi l^2} \log \frac{64e\pi l^6}{1-\epsilon} \right. \\
&\quad \left. - q \log e \left(1 - (1-\epsilon) \left(1 + \frac{1}{2l^2} + \frac{1}{\sqrt{2}l} \right)^2 \right) \right) n
\end{aligned}$$

□

Proof of Theorem 2. We assume

$$n \geq \frac{1000}{9} \left(\frac{\log \left(\frac{1-p_{\min}}{p_{\min}} \frac{p_{\max}}{1-p_{\max}} \right) + 1}{\delta(1-\delta)p_{\min}\epsilon} \right)^2$$

For any S , with $S \subseteq R$, define the binary function χ as follows

$$\chi_{\epsilon,l}(S) = \begin{cases} 1 & \text{if } S \text{ has an } l\text{-cover of cost at most } (1-\epsilon)n/\pi \\ 0 & \text{otherwise} \end{cases}$$

Clearly, it holds that

$$\Pr[R, \epsilon, l] = \sum_{S \subseteq R} P_R(S) \chi_{\epsilon,l}(S) \quad (7)$$

Let us partition R into t regions R_1, R_2, \dots, R_t where $|R_j| = n_j$ such that for $1 \leq j \leq t$

$$\left\lfloor \frac{n}{t} \right\rfloor \leq n_j \leq \left\lceil \frac{n}{t} \right\rceil \text{ and } R_j = \left\{ \sum_{i=1}^{j-1} n_i + k \mid 1 \leq k \leq n_j \right\}.$$

Define μ_j as the expected number of points in j , i.e., $\mu_j = \sum_{i \in R_j} p_i$. Let \mathcal{F} be the family of subsets of R having, in each region, a number of points not too small w.r.t. the expected number, i.e.,

$$\mathcal{F} = \{S \in 2^R \mid |S \cap R_j| \geq (1-\delta)\mu_j, 1 \leq j \leq t\}$$

From (7) we get

$$\Pr[R, \epsilon, l] = \sum_{S \subseteq \mathcal{F}} P_R(S) \chi_{\epsilon,l}(S) + \sum_{S \subseteq 2^R - \mathcal{F}} P_R(S) \chi_{\epsilon,l}(S) \quad (8)$$

We start giving an upper bound on the first addend of the right-hand of the above equation. Let

$$A = \{\vec{k} = (k_1, \dots, k_t) \mid \vec{k} \in \mathcal{Z}^t \text{ and } (1-\delta)\mu_j \leq k_j \leq n_j, 1 \leq j \leq t\}$$

and, for each $\vec{k} \in A$, define

$$\mathcal{F}_{\vec{k}} = \{S \in \mathcal{F} \mid |S \cap R_j| = k_j, 1 \leq j \leq t\}$$

Consider any set $S_{\vec{k}} \in \mathcal{F}_{\vec{k}}$ such that $P_R(S_{\vec{k}}) \geq P_R(S)$ for every $S \in \mathcal{F}_{\vec{k}}$. Then,

$$\begin{aligned} \sum_{S \subseteq \mathcal{F}} P_R(S) \chi_{\epsilon,l}(S) &= \sum_{\vec{k} \in A} \sum_{S \in \mathcal{F}_{\vec{k}}} P_R(S) \chi_{\epsilon,l}(S) \leq \\ &\sum_{\vec{k} \in A} \sum_{S \in \mathcal{F}_{\vec{k}}} P_R(S_{\vec{k}}) \chi_{\epsilon,l}(S) = \sum_{\vec{k} \in A} P_R(S_{\vec{k}}) \sum_{S \in \mathcal{F}_{\vec{k}}} \chi_{\epsilon,l}(S) \\ &\leq \sum_{\vec{k} \in A} P_R(S_{\vec{k}}) 2^{\lambda(n, \min\{\frac{k_j}{n_j} \mid 1 \leq j \leq t\}, \epsilon, l, t)} \prod_{i=1}^t \binom{n_i}{k_i} \end{aligned} \quad (9)$$

where the last step follows from Lemma 3. Function λ is decreasing in q and

$$\begin{aligned} \min_{1 \leq j \leq t} \left\{ \frac{k_j}{n_j} \right\} &\geq \min_{1 \leq j \leq t} \left\{ \frac{(1-\delta)\mu_j}{n_j} \right\} \geq \min_{1 \leq j \leq t} \left\{ \frac{(1-\delta)p_{\min}n_j}{n_j} \right\} \\ &= (1-\delta)p_{\min} \end{aligned}$$

We thus get

$$\begin{aligned} \sum_{S \subseteq \mathcal{F}} P_R(S) \chi_{\epsilon,l}(S) &\leq \sum_{\vec{k} \in A} P_R(S_{\vec{k}}) 2^{\lambda(n, (1-\delta)p_{\min}, \epsilon, l, t)} \prod_{i=1}^t \binom{n_i}{k_i} = \\ &= 2^{\lambda(n, (1-\delta)p_{\min}, \epsilon, l, t)} \sum_{\vec{k} \in A} P_R(S_{\vec{k}}) \prod_{i=1}^t \binom{n_i}{k_i} = \\ &= 2^{\lambda(n, (1-\delta)p_{\min}, \epsilon, l, t)} \sum_{\vec{k} \in A} \sum_{S \in \mathcal{F}_{\vec{k}}} P_R(S_{\vec{k}}) \end{aligned} \quad (10)$$

Assume without loss of generality that the points in R are numbered in increasing order w.r.t. their probability i.e. $0 < p_{\min} = p_1 \leq p_2 \leq \dots \leq p_n = p_{\max} < 1$. Let $q_0 = p_{\min} \leq q_1 \leq q_2 \leq \dots \leq q_t = p_{\max}$ where $q_j = \max\{p_i \mid i \in R_j\}$, for $1 \leq j \leq t$. Consider any set $S \in \mathcal{F}_{\vec{k}}$, then

$$\begin{aligned} P_R(S_{\vec{k}}) &= \prod_{j=1}^t P_{R_j}(S_{\vec{k}} \cap R_j) \\ &= \prod_{j=1}^t \left(\prod_{i \in S_{\vec{k}} \cap R_j} p_i \prod_{i \in R_j - S_{\vec{k}}} (1-p_i) \right) = \prod_{j=1}^t \left(\prod_{i \in S \cap R_j} p_i \prod_{i \in R_j - S} (1-p_i) \right) \\ &\leq \prod_{j=1}^t \left(P_{R_j}(S \cap R_j) \prod_{i \in (S - S_{\vec{k}}) \cap R_j} \frac{1-q_{j-1}}{q_{j-1}} \prod_{i \in (S_{\vec{k}} - S) \cap R_j} \frac{q_j}{1-q_j} \right) \\ &= \prod_{j=1}^t \left(P_{R_j}(S \cap R_j) \left(\frac{1-q_{j-1}}{q_{j-1}} \right)^{|(S - S_{\vec{k}}) \cap R_j|} \left(\frac{q_j}{1-q_j} \right)^{|(S_{\vec{k}} - S) \cap R_j|} \right) \\ &\leq \prod_{j=1}^t \left(P_{R_j}(S \cap R_j) \left(\frac{1-q_{j-1}}{q_{j-1}} \frac{q_j}{1-q_j} \right)^{n_j} \right) \\ &\leq \prod_{j=1}^t P_{R_j}(S \cap R_j) \prod_{j=1}^t \left(\frac{1-q_{j-1}}{q_{j-1}} \frac{q_i}{1-q_j} \right)^{\lceil \frac{n}{t} \rceil} \\ &= P_R(S) \left(\frac{1-p_{\min}}{p_{\min}} \frac{p_{\max}}{1-p_{\max}} \right)^{\lceil \frac{n}{t} \rceil} \end{aligned}$$

(10) implies that

$$\begin{aligned} \sum_{S \subseteq \mathcal{F}} P_R(S) \chi_{\epsilon,l}(S) &\leq 2^{\lambda(n, (1-\delta)p_{\min}, \epsilon, l, t)} \sum_{\vec{k} \in A} \sum_{S \in \mathcal{F}_{\vec{k}}} P_R(S) \left(\frac{1-p_{\min}}{p_{\min}} \frac{p_{\max}}{1-p_{\max}} \right)^{\lceil \frac{n}{t} \rceil} \\ &= 2^{\lambda(n, (1-\delta)p_{\min}, \epsilon, l, t)} \left(\frac{1-p_{\min}}{p_{\min}} \frac{p_{\max}}{1-p_{\max}} \right)^{\lceil \frac{n}{t} \rceil} \cdot \sum_{S \in \mathcal{F}} P_R(S) \\ &\leq 2^{\lambda(n, (1-\delta)p_{\min}, \epsilon, l, t)} \left(\frac{1-p_{\min}}{p_{\min}} \frac{p_{\max}}{1-p_{\max}} \right)^{\lceil \frac{n}{t} \rceil} \end{aligned} \quad (11)$$

Now we provide an upper bound on

$$\lambda(n, (1-\delta)p_{\min}, \epsilon, l, t) = (f(l) - g(l))n + \log(4n) + t$$

where

$$f(x) = \frac{1-\epsilon}{\pi x^2} \log \frac{64e\pi x^6}{1-\epsilon}$$

and

$$g(x) = -(1-\delta)p_{min} \log e \left(1 - (1-\epsilon) \left(1 + \frac{1}{2x^2} + \frac{1}{\sqrt{2}x} \right)^2 \right)$$

Function $f(x)$ is decreasing for $x \geq \frac{5(1-\epsilon)^{\frac{1}{6}}}{(1-\delta)p_{min}\epsilon}$, so it holds that

$$f(l) \leq \frac{(1-\epsilon)^{\frac{2}{3}}}{\pi} \left(\frac{(1-\delta)p_{min}\epsilon}{5} \right)^2 \log \left(64e\pi \left(\frac{5}{(1-\delta)p_{min}\epsilon} \right)^6 \right)$$

Moreover, for every $a, c > 0$, it holds that $a \log \frac{c}{a} \leq \frac{c}{e} \log e$. Thus, by setting $a = \frac{\delta p_{min}\epsilon}{5}$ and $c = (64e\pi)^{\frac{1}{6}}$, we get

$$f(l) \leq \frac{12}{5(e\pi)^{\frac{5}{6}}} (1-\delta)p_{min}\epsilon \log e \quad (12)$$

Function $g(x)$ is increasing for $x > 1$ and, by a simple calculus, we obtain

$$g(l) \geq \frac{5-\sqrt{2}-1}{5} (1-\delta)p_{min}\epsilon \log e \quad (13)$$

From (12) and (13), we obtain

$$\lambda(n, (1-\delta)p_{min}, \epsilon, l, t) \leq -\frac{7}{100} (1-\delta)p_{min}\epsilon n \log e + \log(4n) + t$$

Moreover since

$$\left\lceil \frac{n}{t} \right\rceil \log \left(\frac{1-p_{min}}{p_{min}} \frac{p_{max}}{1-p_{max}} \right) < \frac{6}{100} (1-\delta)p_{min}n\epsilon$$

(11) implies

$$\sum_{S \subseteq \mathcal{F}} P_R(S) \chi_{\epsilon, l}(S) \leq 2^{-\frac{1}{100} (1-\delta)p_{min}\epsilon n \log e + \log(4n) + t} \quad (14)$$

We now give an upper bound on the second addend in (8). Let $\mu = \min\{\mu_1, \mu_2, \dots, \mu_t\} \geq p_{min} \lfloor \frac{n}{t} \rfloor$, then

$$\begin{aligned} \sum_{S \subseteq 2^R - \mathcal{F}} P_R(S) \chi_{\epsilon, l}(S) &\leq \sum_{S \subseteq 2^R - \mathcal{F}} P_R(S) \\ &= 1 - \sum_{S \subseteq \mathcal{F}} P_R(S) \\ &= 1 - \prod_{j=1}^t P_R(|S \cap R_j| \geq (1-\delta)\mu_j) \\ &= 1 - \prod_{j=1}^t (1 - P_R(|S \cap R_j| < (1-\delta)\mu_j)) \\ &\leq 1 - \prod_{j=1}^t \left(1 - e^{-\frac{\delta^2}{2}\mu_j} \right) \\ &\leq 1 - \prod_{j=1}^t \left(1 - e^{-\frac{\delta^2}{2}\mu} \right) \\ &= 1 - \left(1 - e^{-\frac{\delta^2}{2}\mu} \right)^t \\ &\leq t e^{-\frac{\delta^2}{2}\mu} e^{t e^{-\frac{\delta^2}{2}\mu}} \\ &= e^{\ln t - \frac{\delta^2}{2}\mu + t e^{-\frac{\delta^2}{2}\mu}} \\ &\leq e^{t - \frac{\delta^2}{2}\mu} \\ &\leq e^{-\frac{\delta^2}{2} p_{min} \lfloor \frac{n}{t} \rfloor + t} \end{aligned} \quad (15)$$

Finally, by combining (8), (14) and (15), the theorem follows. \square

3. AN ALMOST OPTIMAL SOLUTION

We now provide an efficient construction of a covering range assignment for a random set $S \subseteq R$ of energy cost very close to the lower bound $(1-\epsilon)n/\pi$. Then we will transform it, with additional cost $o(n)$ only, into a feasible broadcast range assignment that uses $\Theta(\log(n/\log n))$ ranges and such that the (positive) smallest among them, i.e. l_1 , is $\Theta(\sqrt{\log n})$.

The disk covering construction. The construction of the covering is recursive and exploits a tiling of the square with octagons and triangles.

The square R of side \sqrt{n} is partitioned into four triangles and an octagon (see Figure 1); up to when there exists a triangle with side $c\sqrt{\log n}$, it is further on partitioned into five triangles (three small and two big triangles) and an octagon (see Figure 2).

Starting from this partition, it is possible to produce a *disk* covering *OCT* as follows (in the sequel, a range assignment is seen as a *disk* assignment with centers on nodes in S):

- for each triangle of the partition, if it contains at least one node, then one of them is selected as center of a disk having radius $c\sqrt{2\log n}$. Observe that this disk covers any other point inside the same triangle.
- for each octagon, if it contains at least one point that is not covered yet, then Lemma 5 implies that there

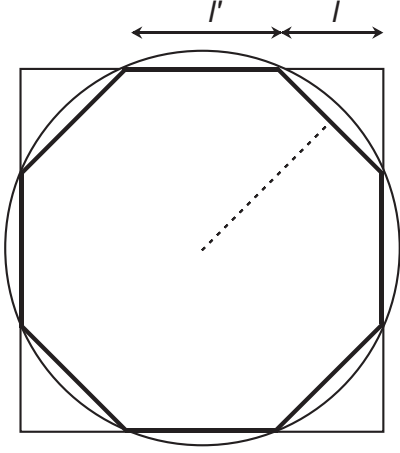


Figure 1: The starting configuration of the partition of the square into triangles and an octagon.

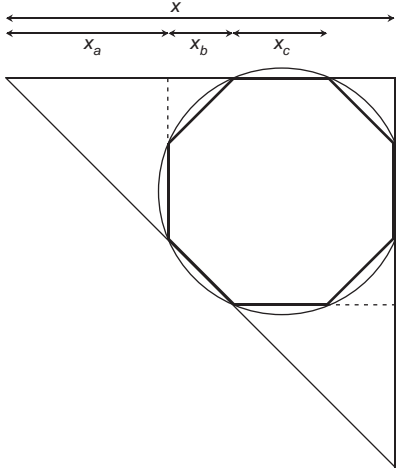


Figure 2: The general step of the partition procedure of the square into triangles and octagons.

is a node at distance $O(\sqrt{\log n})$ to the center of the octagon, w.h.p. Let this node be the center of a disk having radius $r + O(\sqrt{\log n})$, where r is the radius of the disk that circumscribes the considered octagon; the introduced disk covers all points in the octagon.

Theorem 4 *Given a random set S , then, w.h.p., disk covering OCT has cost*

$$\text{cost}(\text{OCT}) \leq 1, 12 \frac{n}{\pi} + o(n)$$

PROOF. Let S be the set of all the octagons in the partition, and for each $s \in S$ call r_s the radius of the disk that circumscribes octagon s . Denoting by t be the number of triangles in the partition, by construction it holds that:

$$\text{cost}(\mathcal{C}) \leq 2tc^2 \log n + \sum_{s \in S} (r_s + c\sqrt{\log n})^2 \quad (16)$$

Let l be the side of the triangles created during the first step and let l' be the side of the first octagon (see Figure 1). The

following equations hold: $2l + l' = \sqrt{n}$ and $l' = \sqrt{2}l$. From these, we derive:

$$l = \frac{\sqrt{n}}{2+\sqrt{2}} \quad l' = \frac{\sqrt{2n}}{2+\sqrt{2}} \quad (17)$$

The recursive step depicted in Figure 2 produces triangles of two different sides and an octagon. Let x_a, x_b and x_c be the lengths of the sides of the bigger triangles, the smaller triangles and the octagon, respectively. These lengths are tied from the following relationships: $x_c = \sqrt{2}x_b$, $x_a = x_b + x_c$ and $x_a + 2x_b + x_c = l$, implying:

$$x_a = \frac{l}{\sqrt{2}+1} \quad x_b = \frac{l}{(\sqrt{2}+1)^2} \quad x_c = \frac{\sqrt{2}l}{(\sqrt{2}+1)^2} \quad (18)$$

From (17) and (18), the triangles of the partition, generated during step i , have side length $x_i = \frac{\sqrt{2n}}{(2+\sqrt{2})(\sqrt{2}+1)^i}$, where $0 \leq i < k$ and k is the smallest integer value such that $x_i < c\sqrt{\log n}$, i.e.,

$$k = \left\lceil \frac{1}{2} \log_{(1+\sqrt{2})} \frac{n}{c^2(2+\sqrt{2})^2 \log n} \right\rceil \quad (19)$$

Observe that all octagons (but the first one) of the partition are produced by partitioning some triangle of side length x_i , $0 \leq i < k$. Denote by r the radius of the disk that circumscribes the first octagon, by r_i the radius of the disk that circumscribes the octagon produced by partitioning a triangle of side length x_i , and by t_i the number of such triangles. Then, we can rewrite (16) as follows:

$$\begin{aligned} \text{cost}(\mathcal{C}) = t \cdot o(\log n) + \sum_{j=0}^{q-1} \sum_{i=0}^{k-1} t_i (r_i^2 + 2c\sqrt{\log n} r_i + c^2 \log n) \\ + (r + c \log n)^2 \end{aligned} \quad (20)$$

We remind that the radius of the disk that circumscribes a regular octagon having side l is $\frac{l}{\sqrt{2}-\sqrt{2}}$. So, we can use (17) and (18) to compute the following values of r and r_i , respectively, where $0 \leq i < k$:

$$r = \frac{\sqrt{2n}}{(2+\sqrt{2})\sqrt{2}-\sqrt{2}} \quad r_i = \sqrt{\frac{2n}{2-\sqrt{2}}} \frac{1}{(\sqrt{2}+1)^{i+2}} \frac{1}{2+\sqrt{2}} \quad (21)$$

In order to compute the value of t_i , observe that trivially $t_0 = 4$ (see Figure 1) and $t_1 = 8$ (see Figure 2). At step i , $t_i = 2t_{i-1} + 3t_{i-2}$. Unrolling the recursion we get:

$$t_i = 3^{i+1} + (-1)^i \leq 3^{i+1} + 1 \quad (22)$$

In order to evaluate $\text{cost}(\mathcal{C})$, we bound all terms appearing in (20) by exploiting (21) and (22):

$$\begin{aligned} \sum_{i=0}^{k-1} t_i r_i^2 &< \frac{n}{(2+\sqrt{2})(\sqrt{2}+1)^4} \sum_{i=0}^{+\infty} \frac{3^{i+1} + 1}{(\sqrt{2}+1)^{2i}} \\ &= \frac{n}{(2+\sqrt{2})(\sqrt{2}+1)^4} \frac{(\sqrt{2}+1)(4\sqrt{2}+3)}{2\sqrt{2}} \\ &< \frac{(64-45\sqrt{2})n}{4\sqrt{2}} \end{aligned} \quad (23)$$

$$(r + c \log n)^2 = \frac{(2-\sqrt{2})n}{2} + o(\sqrt{n \log n}) \quad (24)$$

By combining Equations (22) and (19) we obtain:

$$\begin{aligned} t &= 3^{k+1} + 1 \\ &\leq 9 \left(\frac{1}{c(2 + \sqrt{2})} \right)^{\frac{1}{\log_3(\sqrt{2}+1)}} \left(\frac{n}{\log n} \right)^{\frac{1}{2 \log_3(\sqrt{2}+1)}} \\ &= o \left(\frac{n}{\log n} \right) \end{aligned} \quad (25)$$

where the last step is true because $\frac{1}{2 \log_3(\sqrt{2}+1)} < 1$. Furthermore,

$$2c\sqrt{\log n} \sum_{i=0}^{k-1} t_i r_i = o(n) \quad (26)$$

Equation (22) implies that $\sum_{i=0}^{k-1} t_i < t^k$; Then, from (25) we get:

$$c^2 \log n \sum_{i=0}^{k-1} t_i = o(n) \quad (27)$$

By combining formulas (20), (23), (24), (26) and (27) we conclude that

$$\text{cost}(\mathcal{C}) = \frac{(64 - 45\sqrt{2})n}{4\sqrt{2}} + \frac{(2 - \sqrt{2})n}{2} + o(n) < 1.1204 \frac{n}{\pi} + o(n) \quad \square$$

From Covering to Broadcasting. In order to guarantee that the produced covering becomes a broadcast, we need to connect the source to the disk centers in *OCT*. We start from the source, located in any place of the square, and build a chain of disks towards the center of the grid. Thanks to Lemma 5, the maximum radius of such disks can be bounded by $O(\sqrt{\log n})$, w.h.p. (see Fig. 3). We now show that the additional cost due to this construction turns out to be sub-linear.

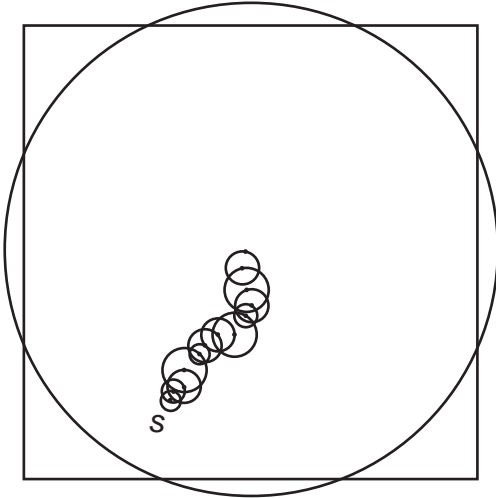


Figure 3: Construction of the chain of disks connecting the source to the center of the first disk.

The cost of the connection between the source and the center of the square is $O(\sqrt{n}\sqrt{\log n})$ w.h.p. Then we have to

connect all the other centers to points already reached by the information sent from the source. The total cost due to this step is bounded by $\sum_{j=1}^k t_j x_j O(\sqrt{\log n})$. By replacing the formulas for t_j and x_j we get:

$$\begin{aligned} \sum_{j=1}^k t_j x_j O(\sqrt{\log n}) &= O(\sqrt{\log n}) \sum_{j=1}^k (3^{j+1} + 1)(\sqrt{2} - 1)^j l_0 \\ &= O(\sqrt{\log n}) \frac{\sqrt{n}}{2 + \sqrt{2}} \left(\sum_{j=1}^k 3^{j+1} (\sqrt{2} + 1)^j + \sum_{j=1}^k (\sqrt{2} - 1)^j \right) \\ &= O(\sqrt{\log n}) \frac{\sqrt{n}}{2 + \sqrt{2}} \Theta((3(\sqrt{2} - 1))^{k+1}) \\ &= O(\sqrt{\log n}) \frac{\sqrt{n}}{2 + \sqrt{2}} \left(\frac{1}{c(2 + \sqrt{2})} \sqrt{\frac{n}{\log n}} \right)^{\frac{1}{\log_3(\sqrt{2}-1)(\sqrt{2}+1)}} \end{aligned}$$

This cost is sub-linear since it is $O(n^{0.63})$. It is not hard to verify that the above overall construction can be performed in $O(n \log n)$ time.

4. AN EFFICIENT DISTRIBUTED PROTOCOL

Let us consider the following simple algorithm to construct a broadcast range assignment. Let l be *any* range in Γ such that $l \geq 2\sqrt{2}c\sqrt{\log n}$ where c is the constant determined by Lemma 5 below.

Algorithm CELL-ALG.

- Grid R is partitioned into *square cells* of side length $\lambda = l/(2\sqrt{2})$.
- In every non-empty cell, choose one of its nodes and assign range l to it. This node is called the *pivot* of the cell.
- The cell containing the source will have the source as pivot.
- All other nodes have range 0.

The proof of the following lemma is a simple application of Chernoff's Bound.

Lemma 5 *Let p_{\min} , p_{\max} , and c be three constants such that $0 < p_{\min} \leq p_{\max} < 1$ and $c \geq 16/p_{\min}$. Let $S \subseteq R$ be a random grid. Consider the partition of R into square cells of side length λ where $c\sqrt{\log n} \leq \lambda \leq \sqrt{n}$. Then, a constant $\gamma > 0$ exists such that every cell contains w.h.p. at least $\gamma\lambda^2$ nodes. Constant γ can be set as $(1/2)p_{\min}$.*

It is then easy to prove the following

Theorem 6 *Algorithm CELL-ALG yields a broadcast range assignment r that is w.h.p. feasible and its cost satisfies*

$$\text{cost}(r) = \frac{n}{\lambda^2} \cdot (2\sqrt{2}\lambda)^2 = 8n$$

Thanks to our lower bound in Theorem 2, CELL-ALG yields *constant* approximation.

Making it in distributed way. Algorithm CELL-ALG can be converted, *without paying any extra energy cost*, into an efficient, *energy-load balanced* protocol that performs a *sequence of broadcast operations*. We describe the protocol for the many-broadcast scenario and, thus, besides minimizing the energy spent by a single broadcast operation, we aim to evenly distribute the transmission task among all nodes (but the source).

According to the standard radio communication model [2, 11, 26], we assume that nodes act in discrete uniform *time steps* and are non spontaneous. However, we assume a weaker, *local synchronous* model: if, at a given time step t , the range of a message transmission covers a cell, then, at time step $t + 1$, (only) the nodes of that cell are activated and, so, they will agree on the same time step. We assume that every node v knows the number n of points and its relative coordinates in the square grid R . From its relative coordinates every node computes a unique *local* label with respect to its cell. These local labels vary from 1 to λ^2 . The k -th message sent by the source is denoted as M_k . *Phase k* consists of the sequence of time steps where M_k is broadcasted. We assume that M_k contains the value k .

The protocol performs, in parallel, two tasks: *i*) it constructs a broadcast communication graph starting from the source and *ii*) transmits the source message along this graph to all nodes. The procedure is executed for every broadcast operation from source s . Every node keeps a local counter **counter** initially set to -1 .

Procedure BROADCAST(M_k)

Source s transmits, with range l , $\langle M_k, i \rangle$ where i is the index of its cell.

All nodes (but s):

- If $(k \leq \gamma\lambda^2)$ then (γ is the constant of Lemma 5)
 - When a node v receives, for the first time w.r.t. phase k , $\langle M_k, i \rangle$ from the pivot of a neighbor cell i , it becomes active.
 - An active node, at every time step, increments its local counter **counter** by one and checks whether its local label is equal to the value of its **counter**. If this is the case, it becomes the pivot of its cell and transmits, with range l , $\langle M_k, j \rangle$ where j is the index of its cell.
 - When an active node in cell i receives $\langle M_k, i \rangle$, it (so the pivot as well) records in a *local array* $P[k]$ the current value of its **counter**, i.e. the local label of the pivot, and becomes inactive.
- else (i.e. $(k > \gamma\lambda^2)$)
 - When a node v receives, for the first time w.r.t. phase k , $\langle M_k, i \rangle$ from the pivot of a neighbor cell i , it checks if its local label is equal to $P[k \bmod \gamma\lambda^2]$. If this is the case, it becomes the pivot of its cell and transmits, with range l , $\langle M_k, j \rangle$ where j is the index of its cell.

Fact 7 *Even though nodes initially do not know anything about each other, all nodes in the same cell are activated (and deactivated) at the same time step; so, their local counters share the same value at every time step. Furthermore, after the first $\gamma\lambda^2$ broadcast operations (i.e. phases), all nodes in the same cell know the set P of pivots of that cell.*

More precisely, if $j_0 < j_1 < j_2 < \dots j_k \dots$ are the local labels of the nodes in a cell, then, during the first $\gamma\lambda^2$ broadcast operations (i.e. phases), the pivot of the cell at phase k will be the node having local label j_k .

Procedure BROADCAST has the following properties.

Energy Cost. As for each single broadcast operation, BROADCAST yields a broadcast range assignment equivalent to that of CELL-ALG. So, Theorem 6 holds as well.

Work Complexity.

Definition 8 *Let $\{G_1, G_2, \dots, G_h\}$ be the set of all messages sent by the nodes according to a protocol P . Then, the work complexity of P is*

$$\sum_{i=1}^h l_i^2, \text{ where } l_i \text{ is the range used to send } G_i$$

The overall number of node transmissions (i.e. the message complexity) of every execution of BROADCAST is $8n/l^2$. Each transmission has range l , so the work complexity is not larger than $8n$.

As for the many-broadcast scenario, our lower bound in Theorem 2 easily implies that a work $k(1 - \epsilon)(n/\pi)$ is w.h.p. required to perform a sequence of k broadcasts (since the lower bound holds for the energy cost). It follows that our protocol achieves an almost optimal work complexity for the many-broadcast operation as well.

Load Balancing and Network Lifetime. The *expensive* pivot's task is evenly assigned, w.h.p., to $\gamma\lambda^2$ nodes (see Lemma 5) in the same cell by using a round robin schedule. This is crucial when the number of broadcasts increases and nodes have limited battery charge. As for the many-broadcast operation, it is possible to show that our protocol achieves an almost maximal lifetime according to the consumption model in [7, 6]. In this model, the goal is to maximize the lifetime of the network while guaranteeing, at any phase k , a broadcast operation from the source. Formally, each node v is initially equipped with a battery charge³ $B > 0$. Whenever a node transmits with range l , its battery charge is reduced by amount $\beta \cdot l^2$ where l denotes the range assigned to node v and $\beta > 0$ is a fixed constant depending on the adopted technology. We assume $\beta = 1$, however, all our results holds for any $\beta > 0$.

Then, the MAX LIFETIME problem is to maximize the number of independent broadcast operations till some node will die (i.e. its battery charge becomes 0). In [7], MAX LIFETIME is shown to be NP-hard.

Theorem 9 *BROADCAST performs a sequence of independent broadcast operations whose length is only a constant factor smaller than the optimum, w.h.p.*

Sketch of proof. We have already observed that the work complexity of BROADCAST for any single broadcast operation is not larger than αopt , where α is a positive constant

³Here we assume that, at the very beginning, all nodes are in the same energy situation.

and opt is the optimal work complexity. So, the maximal number of independent broadcast operations is not larger than nB/opt . Thanks to the local round robin strategy in every cell, the energy load of the many-broadcast operation is well balanced over at least a (large) constant fraction η of all nodes. So the number of broadcast operations performed by BROADCAST is at least $\frac{\eta n B}{\alpha_{\text{opt}}} = (\eta/\alpha) \frac{nB}{\text{opt}}$, w.h.p. \square

(Amortized) Completion Time.

Theorem 10 *The amortized completion time (i.e. the average number of time steps to perform one broadcast operation) over a sequence of T broadcast operations is w.h.p.*

$$O(l\sqrt{n}/T + \sqrt{n}/l)$$

Sketch of proof. For a single broadcast operation performed by BROADCAST, we define the *delay* of a cell as the number of time steps from its activation time till the selection of its pivot. Observe that the sum of delays introduced by a cell during the first $\gamma\lambda^2$ broadcasts is at most $\lambda^2 = \Theta(l^2)$. Then, the delay of any cell becomes 0 for all broadcasts after the first $\gamma\lambda^2$ ones. Moreover, a broadcast can pass over at most $O(\sqrt{n}/l)$ cells. By assuming that a maximal length path (this length being $\Theta(\sqrt{n}/l)$) together with maximal cell delay can be found in each of the first $\min\{\gamma\lambda^2, T\}$ broadcasts, we can bound the maximal overall delay with

$$O(l\sqrt{n}) \quad (28)$$

Finally, the number of time steps required by every broadcast without delays is

$$O(\sqrt{n}/l) \quad (29)$$

since the length of any path on the broadcast tree is $O(\sqrt{n}/l)$. By combining (28) and (29), we get the theorem bound. \square

For brevity's sake, the amortized completion time has been analyzed without considering the *interferences* due to *collisions* among pivot transmissions [2]. However, in order to avoid such collisions, we can further organize BROADCAST into iterative *stages*: in every stage, only cells with not colliding pivot transmissions are active. Since the number of cells that can interfere with a given cell is constant, this further scheduling will increase the overall time by a constant factor only. This iterative process can be efficiently performed in a distributed way since every node knows n and its position, so it knows its cell.

Corollary 11 *The completion time of one single broadcast operation is $O(l\sqrt{n})$.*

The worst scenario for our protocol occurs when T is small, say $T = O(1)$. Indeed, assume that a transmission range $l = \Theta(\sqrt{\log n})$ is available in Γ , then we get an amortized completion time $O(\sqrt{n \log n})$ that is a factor $\log n$ larger than the optimum. Notice that in this case, the network diameter is $\Theta(\sqrt{n/\log n})$ w.h.p.

Whenever $T = \Omega(\log n)$, we instead get $O(\sqrt{n/\log n})$ amortized completion time which is optimal.

4.1 Experimental results

In this subsection, we present the experimental results we have obtained by running Algorithm CELL-ALG. We have generated 1000 instances for every side length

$$\sqrt{n} \in \{13, 20, 25, 30, 50, 100\}$$

and for *node-probability* $p \in \{0.2, 0.5\}$. As usual, our implementation benefits of some parameter tuning and optimization: the pivot node (but the source node) inside every cell is the one closer to the center of the cell and useless, redundant ranges are removed. These tasks can be performed also by the distributed protocol, after the first phase (i.e. for $t \geq \gamma\lambda^2$), without paying any extra energy cost since, after that time, every node of a cell knows all its cell neighbors. Moreover, the transmission range l is set⁴ to $\sqrt{2} \log n$, while the cell-size parameter λ is set to $\log n$. Notice that, according to such choices, the feasibility (i.e., the existence of a path from the source node to all other nodes in the induced communication graph) is tested too. In Table 1 (columns “# of feasible sol.”), the number of feasible solutions for the different combinations of n and p are reported.

The solution costs of CELL-ALG are compared to the cost of the solution returned by the centralized MST-based algorithm. We remind that while the energy cost of CELL-ALG is an upper bound on the work complexity of our distributed procedure BROADCAST the energy cost of the MST-based solution does not provide any information about the work complexity of its distributed implementations (this can be much larger).

Table 1 shows, for all chosen values of p and \sqrt{n} , the minimum, average and maximum ratio between the costs of the solutions returned by the two algorithms. As for CELL-ALG, only the costs of *feasible* solutions are considered.

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⁴Notice that, for the tested sizes n , this range is smaller than the threshold $2\sqrt{2}c\sqrt{\log n}$ defined in Section 4: this is the reason why the feasibility rate is not 100% for large n .

cost(CELL-ALG) vs cost(MST)								
\sqrt{n}	$p = 0.2$				$p = 0.5$			
	# of feasible sol.	min	average	max	# of feasible sol.	min	average	max
13	744/1000	2.044	2.072	2.118	1000/1000	2.544	2.808	3.086
20	892/1000	1.896	2.092	2.145	999/1000	2.506	2.666	2.994
25	762/1000	2.040	2.127	2.164	998/1000	2.341	2.617	2.808
30	740/1000	2.163	2.217	2.242	997/1000	2.512	2.610	2.666
50	858/1000	2.313	2.398	2.450	999/1000	2.673	2.824	2.949
100	967/1000	2.300	2.347	2.352	1000/1000	2.604	2.659	2.702

Table 1: Experimental results.

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